

Quantum Bell Inequalities from Macroscopic Locality

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We propose a method to generate analytical quantum Bell inequalities based on the principle of Macroscopic Locality. By imposing locality over binary processings of virtual macroscopic intensities, we establish a correspondence between Bell inequalities and quantum Bell inequalities in bipartite scenarios with dichotomic observables. We discuss how to improve the latter approximation and how to extend our ideas to scenarios with more than two outcomes per setting.

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I. INTRODUCTION

Any probability distribution achievable in the measurement of a bipartite quantum system can be written as

$$P(a, b|x, y) = \text{Tr}(\rho \Pi_a^x \otimes \Pi_b^y)$$

where ρ is a density operator of unrestricted dimension and the Π 's are projectors. The set of quantum correlations (or *quantum set*) is the set of all probability distributions that can be written in this form. It has long been known that some correlations in this set are *nonlocal*, in the sense that they cannot be produced by pre-established agreement [1]. On another front, the quantum set is strictly smaller than the *no-signaling set*, i.e. those correlations that cannot be used to send a signal [2].

The mathematical characterization of the quantum set is not trivial [3]. One of the few definite results was obtained for the case of two dichotomic measurements per site, i.e., x, y, a and b can take only two values each (a situation that we shall refer to as 2222). In this scenario, up to symmetries, the only Bell inequality is the Clauser-Horne-Shimony-Holt (CHSH) [4]

$$\sum_{x,y} (-1)^{xy} \langle a_x b_y \rangle \leq 2, \quad (1)$$

where for convenience of notation we have chosen $x, y \in \{0, 1\}$, $a, b \in \{-1, +1\}$. Tsirelson and Landau (and, much later, Masanes) independently proved that the set of quantum correlations is bounded by a *quantum Bell inequality* obtained from CHSH upon replacing $\langle a_x b_y \rangle$ by $\frac{2}{\pi} \arcsin \langle a_x b_y \rangle$ [5–7]. More recently, Navascués, Pironio and Acín proved that, if we relax the tensor structure assumption in (1) and instead demand commutativity between Alice's and Bob's measurement operators, the resulting set of correlations

can be approximated, and ultimately reached, by a hierarchy of semi-definite programs [8, 9]. In the process, they proved that the replacement $\langle a_x b_y \rangle \rightarrow \frac{2}{\pi} \arcsin D_{xy}$, where

$$D_{xy} = \frac{\langle a_x b_y \rangle - \langle a_x \rangle \langle a_y \rangle}{\sqrt{(1 - \langle a_x \rangle^2)(1 - \langle a_y \rangle^2)}} \quad (2)$$

in the CHSH inequality leads to a quantum Bell inequality that is stronger than the previously known one and identifies the set denoted Q^1 , the first step of the semi-definite hierarchy.

Aside from the mathematical line of attack, several authors have followed the spirit of the Popescu-Rohrlich paper [2] and tried to identify *physical principles* that would define the quantum set [10–14]. The principle of information causality has already proved useful to derive bounds on the quantum set in non-trivial scenarios [15]. In this paper, we use the principle of *macroscopic locality* [13] to derive quantum Bell inequalities. The recipe is applicable to all bipartite scenarios. Focusing on *nm22* scenarios, we prove that a quantum Bell inequality can be obtained from any standard Bell inequality through (2). Such inequalities are the only analytical approximations to the quantum set available to date.

II. TOOLS

A. Macroscopic locality

Before presenting our results, we review briefly the idea of macroscopic locality (ML) [13]. We consider a bipartite system, with a source emitting pairs of particles to Alice and Bob. Alice can choose a measurement $x \in X = \{1, \dots, m_A\}$ from a set of m_A possible settings, each producing d_A possible outcomes, denoted as $a \in \mathcal{A} = \{1, \dots, d_A\}$. Similarly, Bob can choose a

measurement y from the set $Y = \{1, \dots, m_B\}$, with d_B outcomes $b \in \mathcal{B} = \{1, \dots, d_B\}$. One calls a *microscopic experiment* one in which single-particle detections are possible. In this case, routine sampling leads to reconstructing $P(a, b|x, y)$. By contrast, a *macroscopic experiment* is defined as follows: in each run, N pairs are sent; Alice measures all her particles with setting x and records the number n_a of particles that produced the outcome $a \in \mathcal{A}$; Bob does similarly. After repeating this procedure several times, one can reconstruct $P(\vec{n}_A, \vec{n}_B|x, y)$, where $\vec{n}_A = [n_{a=1}, \dots, n_{a=d_A}]$ and $\vec{n}_B = [n_{b=1}, \dots, n_{b=d_B}]$. Clearly, some microscopic information is lost as soon as $N > 1$: Alice and Bob do not know which of their outcomes came from the same pair.

The principle of ML demands that, in the limit $N \rightarrow \infty$ and under the assumption that Alice's and Bob's measurement devices cannot detect particle number fluctuations smaller than $O(\sqrt{N})$, the macroscopic statistics should not violate any Bell inequality. The principle is not trivial: for instance, one can easily check that, if the source were a Popescu-Rohrlich (PR) box, the macroscopic statistics would violate CHSH up to 4, just as the microscopic statistic do. In fact, ML is respected if and only if the microscopic statistics $P(a, b|x, y)$ belong to Q^1 [13].

B. Recipe for quantum Bell inequalities

It is now easy to formulate our recipe to generate quantum Bell inequalities. Consider a data processing of the macroscopic variables

$$\vec{n}_A \rightarrow \alpha \in \mathcal{A}', \quad \vec{n}_B \rightarrow \beta \in \mathcal{B}'. \quad (3)$$

This defines a scenario with m_A measurements for Alice, m_B for Bob, $|\mathcal{A}'| = d'_A$ outcomes for Alice and $|\mathcal{B}'| = d'_B$ for Bob – in short, denoted as $m_A m_B d'_A d'_B$ scenario. From ML it follows that, if $P(a, b|x, y)$ belongs to the quantum set, $P(\alpha, \beta|x, y)$ cannot violate any Bell inequality. In other words, for any Bell inequality $\mathcal{I} \leq 0$ in the scenario $m_A m_B d'_A d'_B$, the condition $\mathcal{I}[P(\alpha, \beta|x, y)] \leq 0$ defines a quantum Bell inequality for the microscopic correlations. If in addition $P(\alpha, \beta|x, y)$ can be expressed as a function of the parameters of $P(a, b|x, y)$, an analytical bound on the quantum set is obtained.

Of course, we know already that none of these criteria will come closer to the quantum set than Q^1 does; and in general, the bound may even be less tight than Q^1 (we show an explicit example below). However, to date, this is the only known systematic approach to obtain analytical approximations to the quantum set in any scenario. In what follows, we study some specific

examples of quantum Bell inequalities generated with this recipe.

III. CASE STUDIES

A. Two outcomes: Sign-binning

Let $d_A = d_B = 2$ and for convenience we use $\mathcal{A} = \mathcal{B} = \{-1, 1\}$. Consider first a specific choice of measurements x, y (so, for simplicity, we omit them in the notation). From her macroscopic statistics, Alice can infer the microscopic marginal average

$$\langle a \rangle = \langle \Delta n_A \rangle / N \quad \text{with} \quad \Delta n_A \equiv n_{a=+1} - n_{a=-1}. \quad (4)$$

A natural data processing that gives back binary outcomes ($\mathcal{A}' = \{-1, 1\}$) consists in comparing Δn_A with $\langle a \rangle$ in each run:

$$\text{Sign-binning: } \alpha = \begin{cases} +1 & \text{if } \Delta n_A \geq \langle a \rangle \\ -1 & \text{otherwise} \end{cases}. \quad (5)$$

Bob can of course do the same. Now we want to relate α and β obtained by sign-binning to the properties of the microscopic distribution. Any macroscopic run uses N pairs: denote therefore $\{a^{(1)}, \dots, a^{(N)}\}$ the results Alice would have observed on those N pairs in a microscopic experiment, and similarly $\{b^{(1)}, \dots, b^{(N)}\}$ for Bob. Then we define

$$\alpha = \text{sgn} \left(\sum_{k=1}^N \frac{a^{(k)} - \langle a \rangle}{\sqrt{N}} \right) \equiv \text{sgn}(a'), \quad (6)$$

$$\beta = \text{sgn} \left(\sum_{k=1}^N \frac{b^{(k)} - \langle b \rangle}{\sqrt{N}} \right) \equiv \text{sgn}(b'). \quad (7)$$

In the limit $N \rightarrow \infty$, due to the central limit theorem, the joint probability distribution of the variables (a', b') is a gaussian distribution $G_\Gamma(a', b')$ with zero mean and covariance matrix, Γ determined by the microscopic distribution [13]

$$\Gamma = \begin{pmatrix} 1 - \langle a \rangle^2 & \langle ab \rangle - \langle a \rangle \langle b \rangle \\ \langle ab \rangle - \langle a \rangle \langle b \rangle & 1 - \langle b \rangle^2 \end{pmatrix}. \quad (8)$$

Then, with the notation $\bar{f} = \int da' db' G_\Gamma(a', b') f(a', b')$, a straightforward integration shows that

$$\bar{\alpha} = \bar{\beta} = 0, \quad \overline{\alpha\beta} = \frac{2}{\pi} \arcsin D_{xy}. \quad (9)$$

All that we have done is valid for any pair of measurements x, y . Therefore, as claimed, the replacement

D. Binning for $n > 2$ microscopic outcomes.

The sign binning processing is of course a natural processing for binary outcome settings, though not optimal. It can also be applied to settings with more than two outcomes; but here we sketch a more natural processing for such scenarios. As an example, we will focus on the case of 2233: $X = Y = \{0, 1\}$ and $\mathcal{A} = \mathcal{B} = \{0, 1, 2\}$. We first define the quantity

$$I_i^A = \frac{n_i(\vec{a}) - p_i^A N}{\sqrt{N}}, \forall i \in \mathcal{A} \quad (14)$$

for a particular fixed value of x and y . The data processing, in this case called the triangle-binning, is then defined as

$$\alpha = k, \quad \text{s.t.} \quad I_k^A = \max_{i \in \mathcal{A}} \{I_i^A\}. \quad (15)$$

Bob can similarly do the same. In the limit $N \rightarrow \infty$, the probability distribution for the variables (α_x, β_y) is a Gaussian distribution $G_\Gamma(\alpha_x, \beta_y)$ with the covariance matrix Γ defined in the usual sense. The formulas for the probabilities $P(\alpha, \beta|x, y)$ include definite trigonometric integrals; they may not lead to elegant analytical expressions, but they can then be evaluated numerically to check the method. It is known that 2233 has only one relevant Bell inequality of the Collins-Gisin-Linden-Massar-Popescu (CGLMP) type [22]: $I_{\text{CGLMP}} \leq 0$. We can then impose the condition $I_{\text{CGLMP}}[P(\alpha, \beta|x, y)] \leq 0$. The result is shown in Figure 2 where the constrained set of correlation is now denoted as Q^{TB} . Contrary to sign binning, triangle binning does not reproduce the predictions of ML. Triangle binning may thus not be worth pursuing further, but it shows that generalization is indeed possible.

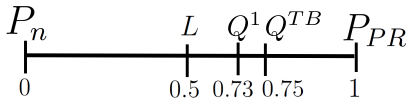


FIG. 2: The line of random marginals defined by the random noise, P_n and the generalized PR box, P_{PR} [23]. The bounds given by the local set, L , macroscopic locality, Q^1 and triangle binning, Q^{TB} are 0.5, 0.75 and 0.73 respectively.

IV. CONCLUSION

In this paper, we have studied a new post-processing method, called sign binning, to define a set of quantum correlations denoted as Q^{SB} , which necessarily contains the quantum set, Q . Sign binning, as mentioned,

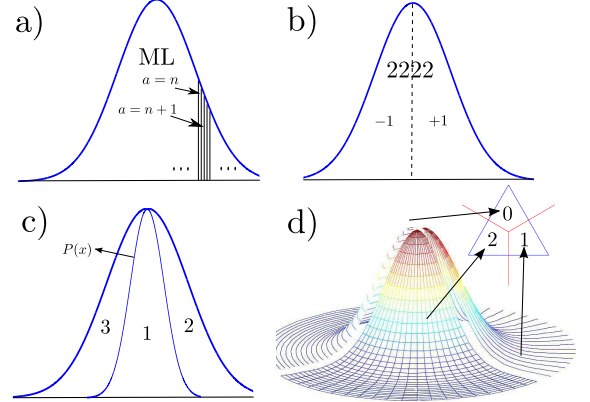


FIG. 3: a) ML in the sense of binning into infinitely many parts. b) Sign binning for scenarios with binary outcomes. c) 3-binning induced by the probability, $P(x)$. d) Triangle-binning for scenarios with three outcomes.

cannot give a bound on the quantum set that is tighter than Q^1 , but it does give a straightforward, intuitive way of generating quantum Bell inequalities. To our knowledge, this recipe is the only method to date to derive analytical quantum Bell inequalities.

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Appendix A: Locality of Sign Binning is Equivalent to Macroscopic Locality in $2n22$

Lemma 1. Let Γ be an $n+2$ square matrix of the form

$$\Gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \quad (A1)$$

where C is a given $2 \times n$ real matrix and A, B are such that $A_{ii} = B_{jj} = 1$ for $i = 1, 2$ and $j = 1, \dots, n$. Then there exists a choice of the remaining entries of A and B such that $\Gamma \geq 0$ iff there exists $x \in [-1, 1]$ such that

$$1 - x^2 - C_{1i}^2 - C_{2i}^2 + 2xC_{1i}C_{2i} \geq 0, \quad (A2)$$

for $i = 1, \dots, n$.

Proof. Let us first prove that, if condition (A2) holds, then Γ can be made positive semidefinite. Suppose

that, indeed, such an x exists and $|x| < 1$. Then, we can take A to be

$$A = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} > 0. \quad (\text{A3})$$

According to Schur's theorem [24], if $A > 0$, a matrix of the form (A1) is positive semidefinite iff $B' \equiv B - C^T A^{-1} C \geq 0$. Since the non-diagonal entries of B are not determined a priori, we can always choose them such that $B'_{ij} = 0$ for $i \neq j$. To see that B' is positive semidefinite, we then only have to show that $B'_{ii} \geq 0$. But

$$B'_{ii} = \frac{1 - x^2 - C_{1i}^2 - C_{2i}^2 + 2xC_{1i}C_{2i}}{1 - x^2}, \quad (\text{A4})$$

that is non-negative by hypothesis. We have just proven that, for $|x| < 1$, condition (A2) grants positive semidefiniteness. Suppose now that (A2) holds for $x = 1$. Then the equation reads

$$-(C_{1i} - C_{2i})^2 \geq 0, \text{ for } i = 1, \dots, n. \quad (\text{A5})$$

It follows that $C_{1i} = C_{2i}$ for all i . In order to show that Γ can be completed to a positive semidefinite matrix, take the orthonormal basis $\{|0\rangle, |1\rangle\}$ and define the vectors:

$$\vec{v}_{1,2} \equiv |0\rangle; \vec{v}_{i+2} \equiv C_{1i}|0\rangle + \sqrt{1 - C_{1i}^2}|1\rangle. \quad (\text{A6})$$

Then, the Gram matrix $\Gamma'_{ij} = \vec{v}_i \cdot \vec{v}_j$ is positive semidefinite, has 1s in the diagonal and its off-diagonal submatrix coincides with C .

The case $x = -1$ can be treated analogously (simply take $\vec{v}_1 = -\vec{v}_2$).

Now we will prove the opposite implication: suppose that there is some way to complete Γ such that $\Gamma \geq 0$. Let $\tilde{\Gamma}$ be such completion and take $x = \tilde{A}_{12}$. If $x = \pm 1$, then the Gram decomposition of $\tilde{\Gamma}_{ij} = \vec{v}_i \cdot \vec{v}_j$ [24] is such that $\|\vec{v}_1\| = \|\vec{v}_2\| = 1$ and $\vec{v}_1 \cdot \vec{v}_2 = \pm 1$. This implies that $\vec{v}_1 = \pm \vec{v}_2$, and so $C_{1i} = \vec{v}_1 \cdot \vec{v}_{2+i} = \pm \vec{v}_2 \cdot \vec{v}_{2+i} = \pm C_{2i}$, and condition (A2) holds for $x = \pm 1$.

Suppose that, on the contrary, $|x| < 1$. Then $A > 0$, so, by Schur's theorem, $\tilde{B}'_{ii} \geq 0$, and condition (A2) holds. \square

Theorem 2. *Let Γ be a matrix such as the one appearing in the definition of the previous lemma. Then, Γ can be made positive semidefinite iff, for all $i, j = 1, \dots, n$, $i \neq j$,*

$$\left| \arcsin(C_{1i}) + \arcsin(C_{2i}) + \arcsin(C_{1j}) - \arcsin(C_{2j}) \right| \leq \pi, \quad (\text{A7})$$

plus permutations of the minus sign.

Proof. By lemma 1, positive semidefiniteness is equivalent to the existence of an $x \in [-1, 1]$ satisfying the conditions (A2). Without loss of generality, we assume that $C_{1i} \equiv \sin(\phi_i)$, $C_{2i} \equiv \sin(\theta_i)$, for $-\pi/2 \leq \theta_i, \phi_i \leq \pi/2$. Then, conditions (A2) can be reexpressed as

$$-\cos(\phi_i + \theta_i) \leq x \leq \cos(\phi_i - \theta_i). \quad (\text{A8})$$

An x satisfying all these conditions exists iff the minimum of the upper limits is greater than or equal to the maximum of the lower limits. In other words, Γ can be completed iff

$$-\cos(\phi_j + \theta_j) \leq \cos(\phi_i - \theta_i), \forall i, j. \quad (\text{A9})$$

Call $\alpha_i \equiv |\phi_i - \theta_i|$, $\beta_j \equiv |\phi_j + \theta_j|$. Then, $0 \leq \alpha_i, \beta_j \leq \pi$, and the positivity condition reads

$$\cos(\alpha_i) + \cos(\beta_j) \geq 0. \quad (\text{A10})$$

Running through all possibilities ($[\alpha_i \leq \pi/2, \beta_j \leq \pi/2]$, $[\alpha_i \leq \pi/2, \beta_j \geq \pi/2]$, $[\alpha_i \geq \pi/2, \beta_j \leq \pi/2]$, $[\alpha_i \geq \pi/2, \beta_j \geq \pi/2]$), one can check that this condition is equivalent to $\alpha_i + \beta_j \leq \pi$, and so we arrive at equations (A7). \square

We will now prove the claimed result in the article that $Q^{SB} = Q^1$ for $2n22$. This proof is an extension of the proof presented in [9]. For the case of $2n22$, let Γ^1 be a certificate of order 1 for a particular $P(a, b|x, y)$. Then we have

$$\Gamma^1 = \begin{pmatrix} 1 & C_A & C_B \\ C_A^T & X & Z \\ C_B^T & Z^T & Y \end{pmatrix}, \quad (\text{A11})$$

where $X_{ii} = Y_{jj} = 1$ for $i = 1, 2, j = 1, \dots, n$. Also, $C_A = (C_{A1}, C_{A2})$ and $C_B = (C_{B1}, \dots, C_{Bn})$ are the marginal correlations for Alice's and Bob's measurement respectively. By Schur's Lemma [24], $\Gamma^1 \geq 0$ is equivalent to the positive semidefiniteness of

$$\bar{\Gamma}^1 = \begin{pmatrix} X & Z \\ Z^T & Y \end{pmatrix} - \begin{pmatrix} C_A^T \\ C_B^T \end{pmatrix} (C_A, C_B) \quad (\text{A12})$$

where the matrix $\bar{\Gamma}^1$ has diagonal elements $\{1 - C_{A1}^2, 1 - C_{A2}^2, 1 - C_{B1}^2, \dots, 1 - C_{Bn}^2\}$. We may assume that all the diagonal elements are non zero for $n \geq 2$; if any of them is zero, then the outcome of that particular measurement is deterministic and can be accounted for with a local hidden variable model. Therefore we can multiply $\bar{\Gamma}^1$ on both sides with the diagonal matrix, $M =$

$\{\sqrt{1 - C_{A1}^2}, \sqrt{1 - C_{A2}^2}, \sqrt{1 - C_{B1}^2}, \dots, \sqrt{1 - C_{Bn}^2}\}.$

The condition $\bar{\Gamma}^1 \geq 0$ is then equivalent to

$$\Gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \geq 0, \quad (\text{A13})$$

where the $2 \times n$ matrix C has elements $C_{ij} = (C_{ij} -$

$C_i C_j) / \sqrt{1 - C_i^2} \sqrt{1 - C_j^2}$ and $A_{ii} = B_{jj} = 1$ for $i = 1, 2$ and $j = 1, \dots, n$. Now by Lemma 1 and Theorem 2, a certificate of order 1 exists, and therefore probability distribution, $P(a, b|x, y)$ is Q^1 if and only if condition in (10) is satisfied.

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